

# Derivation of magnetic Coulomb's law for thin, semi-infinite solenoids

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It is shown that the magnetic force between thin, semi-infinite solenoids obeys a Coulomb-type law, which corresponds to that for magnetic monopoles placed at the end points of each solenoid. We derive the magnetic Coulomb law from the basic principles of electromagnetism, namely from the Maxwell equations and the Lorentz force.

## I. INTRODUCTION

A permanent magnet is an ensemble of microscopic magnetic moments which are oriented along the magnetization direction. A magnetic moment can be modeled as a dipole, i.e., as a slightly displaced pair of magnetic monopoles with opposite polarities. It is analogous to the electric dipole. Another way of modeling is to consider each magnetic moment as a circulating current loop. In terms of far fields, the dipole model and the loop-current model give exactly the same magnetic field. The latter model is more natural because there are no magnetic monopoles found so far and microscopic magnetic moments are always associated with kinetic rotations such as orbital motions or spins of electrons. It also provides correct symmetries with respect to the time and space inversions.

Normally we are interested in macroscopic quantities, which are obtained by coarse-graining of microscopic fields and source distributions<sup>1</sup>. When we coarse-grain the oriented ensemble of microscopic magnetic dipoles in a bar magnet, we have a magnetic north pole at one end and a south pole at the other end as shown in Fig. 1(a). Contributions of the magnetic charges inside are canceled out through the spatial average.

On the other hand, when we coarse-grain the microscopic loop currents, we have macroscopic current which circulates around the bar along the surface as shown in Fig. 1(c). By mimicking the macroscopic current distribution by a coil, we have an electromagnet that is equivalent to the permanent magnet. In this model, the poles, or the ends of bar magnet play no special roles. Again the current model is much more reasonable because of the similarity with the equivalent electromagnet.

The use of the notion of magnetic pole should be avoided as far as possible<sup>2</sup> because of its absence in the framework of electromagnetic theory or in the Maxwell equations. Practically, however, magnetic poles are very convenient to describe the forces between permanent magnets or magnetized objects. The poles are considered as an ensemble of monopoles and the forces between poles are calculated with the magnetic Coulomb law, which is usually introduced just as an analog of the electric Coulomb law or as an empirical rule<sup>5</sup>. For logical consistency, we have to *derive* the magnetic Coulomb law

from Maxwell's equations and the Lorentz force, none of which contain the notion of magnetic monopoles.

The derivation of magnetic Coulomb's law was given more than forty years ago by Chen<sup>3</sup> and Nadeau<sup>4</sup>. In this paper a more detailed analysis based directly on the fundamental laws will be provided. The field singularity for infinitesimal loop currents, which plays crucial roles but was not mentioned in the previous works, will be treated rigorously.

## II. CURRENT DENSITY FOR A THIN SOLENOID

For brevity, we introduce a scalar function  $G_0$  and a vector function  $\mathbf{G}_1$  of position  $\mathbf{r} = (x, y, z)$ :

$$G_0(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}|}, \quad \mathbf{G}_1(\mathbf{r}) = \frac{\mathbf{r}}{4\pi|\mathbf{r}|^3}. \quad (1)$$

We note that  $\nabla G_0 = -\mathbf{G}_1$  and  $\nabla \cdot \mathbf{G}_1 = \delta^3(\mathbf{r})$  hold, where  $\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$  is the three dimensional delta function. With these, the scalar potential for a point charge  $q$  placed at the origin is  $\phi(\mathbf{r}) = (q/\epsilon_0)G_0(\mathbf{r})$ , and the force acting on a charge  $q_1$  at  $\mathbf{r}_1$  from another charge  $q_2$  at  $\mathbf{r}_2$  is  $\mathbf{F}_{1 \leftarrow 2} = (q_1 q_2 / \epsilon_0) \mathbf{G}_1(\mathbf{r}_1 - \mathbf{r}_2)$ . The Biot-Savard law can be expressed as  $d\mathbf{H} = d\mathbf{C} \times \mathbf{G}_1(\mathbf{r})$ , where

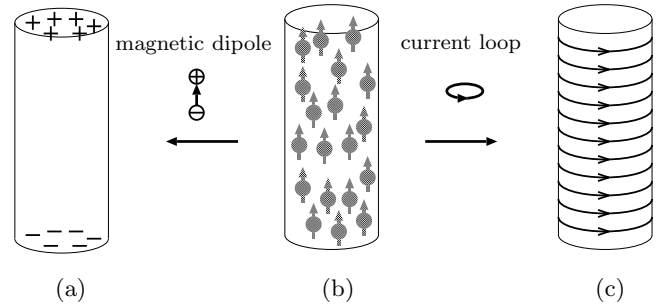


FIG. 1: A permanent magnet is an ensemble of microscopic magnetic moments (b). If we consider each magnetic moment as a magnetic dipole, the corresponding macroscopic picture is two opposite magnetic poles at each end (a). If we adopt the current loop model for magnetic moment, the macroscopic picture consists of circulating current around the side wall (c).

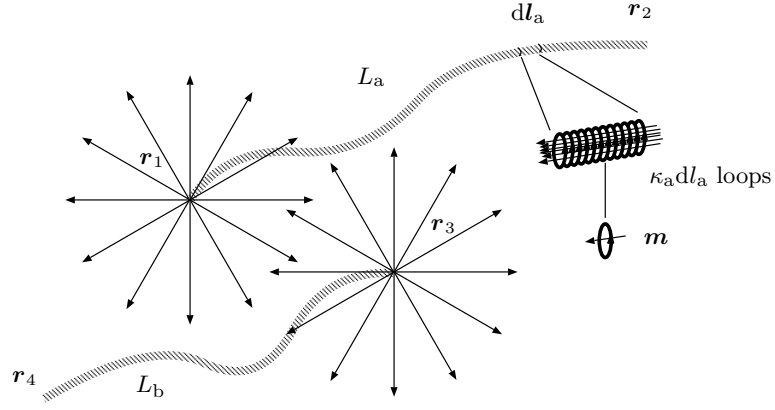


FIG. 2: A pair of thin, semi-infinite solenoids,  $L_a$  and  $L_b$

$d\mathbf{C} = I d\mathbf{l}$  is a current moment (current  $I$  times length  $d\mathbf{l}$ ) located at the origin.

As shown in Fig. 2, a thin solenoid can be constructed as a stack of tiny loop currents at least in principle<sup>6</sup>. The current density for an infinitesimal loop current place at the origin is

$$\mathbf{J}_m(\mathbf{r}) = (-\mathbf{m} \times \nabla) \delta^3(\mathbf{r}) \quad (2)$$

where  $\mathbf{m}$  is the magnetic moment for the loop current and its unit is ( $\text{A m}^2$ ) (See Appendix). To form a solenoid we stack these tiny loop currents along a curve  $L_a$  with a constant line density  $\kappa_a$  (loops per unit length). Each loop are aligned so that the direction of tangent of curve and that of  $\mathbf{m}$  coincide.

The current density distribution for a line segment  $d\mathbf{l}_a$  ( $\parallel d\mathbf{m}$ ) is

$$d\mathbf{J}(\mathbf{r}) = (-C_a d\mathbf{l}_a \times \nabla) \delta^3(\mathbf{r} - \mathbf{r}_a), \quad (3)$$

where  $C_a = \kappa_a m \stackrel{\text{D}}{\sim} \text{A m}$  is the magnetic moment per unit length and characterizes the strength of the solenoid<sup>7</sup>, where  $m = \mathbf{m} \cdot d\mathbf{l}_a / |d\mathbf{l}_a|$  is the magnitude of  $\mathbf{m}$  and  $\mathbf{r}_a$  represents the position of the segment  $d\mathbf{l}_a$ .

### III. FIELD BY A THIN, SEMI-INFINITE SOLENOID

Here we introduce an important formula. For a constant vector  $\boldsymbol{\alpha}$  and a vector field  $\mathbf{V}(\mathbf{r})$ , we have

$$\begin{aligned} (\boldsymbol{\alpha} \times \nabla) \times \mathbf{V} - \boldsymbol{\alpha} \times (\nabla \times \mathbf{V}) \\ = (\boldsymbol{\alpha} \cdot \nabla) \mathbf{V} - \boldsymbol{\alpha} (\nabla \cdot \mathbf{V}). \end{aligned} \quad (4)$$

With the Biot-Savard law and Eq. (2), we can find the strength of magnetic field generated by a magnetic moment  $\mathbf{m}$  placed at the origin as

$$\begin{aligned} \mathbf{H}_m(\mathbf{r}) &= \int dv' \mathbf{J}_m(\mathbf{r}') \times \mathbf{G}_1(\mathbf{r} - \mathbf{r}') \\ &= (-\mathbf{m} \times \nabla) \times \mathbf{G}_1(\mathbf{r}), \end{aligned} \quad (5)$$

where  $dv'$  is a volume element at  $\mathbf{r}'$  and  $\nabla' = \partial/\partial\mathbf{r}'$ . Using Eq. (4), it can be rewritten as

$$\mathbf{H}_m(\mathbf{r}) = -(\mathbf{m} \cdot \nabla) \mathbf{G}_1(\mathbf{r}) + \mathbf{m} \delta^3(\mathbf{r}), \quad (6)$$

where the relations  $\nabla \cdot \mathbf{G}_1(\mathbf{r}) = \delta^3(\mathbf{r})$  and  $\nabla \times \mathbf{G}_1 = 0$  have been used. As shown in Appendix, Eq. (6) can also be derived from the Maxwell equations.

It is well known that the field for an electric dipole moment  $\mathbf{p}$  is  $\mathbf{D}(\mathbf{r}) = -(\mathbf{p} \cdot \nabla) \mathbf{G}_1(\mathbf{r})$ , which contains no delta-function terms unlike Eq. (6). This means that in terms of near fields, the dipole and the current loop do not yield the same field.

From Eq. (6), the strength of the magnetic field created by a line segment  $d\mathbf{l}_a$  is

$$d\mathbf{H}(\mathbf{r}) = (C_a d\mathbf{l}_a \cdot \nabla_a) \mathbf{G}_1(\mathbf{r} - \mathbf{r}_a) + C_a d\mathbf{l}_a \delta^3(\mathbf{r} - \mathbf{r}_a) \quad (7)$$

where  $\nabla_a = \partial/\partial\mathbf{r}_a$ . The integration along a curve  $L_a$  yields the magnetic field created by the solenoid:

$$\begin{aligned} \mathbf{H}(\mathbf{r}) &= \int_{L_a} d\mathbf{H} = C_a [\mathbf{G}_1(\mathbf{r} - \mathbf{r}_1) - \mathbf{G}_1(\mathbf{r} - \mathbf{r}_2)] \\ &\quad + C_a \int_{L_a} \delta^3(\mathbf{r} - \mathbf{r}_a) d\mathbf{l}_a, \end{aligned} \quad (8)$$

where  $\mathbf{r}_2$  and  $\mathbf{r}_1$  are the start and the end points of  $L_a$ . The second term of the right-hand side, which corresponds to the magnetic flux confined in the solenoid, vanishes outside. For semi-infinite cases ( $\mathbf{r}_2 = \infty$ ), we have

$$\mathbf{H}(\mathbf{r}) = C_a \mathbf{G}_1(\mathbf{r} - \mathbf{r}_1) + C_a \int_{L_a} \delta^3(\mathbf{r} - \mathbf{r}_a) d\mathbf{l}_a. \quad (9)$$

The first term of the right-hand side, which represents the magnetic field outside of the solenoid, is equivalent to the field for a monopole  $g_a = \mu_0 C_a$  located at  $\mathbf{r}_1$ :

$$\mathbf{B}(\mathbf{r}) = g_a \mathbf{G}_1(\mathbf{r} - \mathbf{r}_1) = \frac{g_a}{4\pi} \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}. \quad (10)$$

The dimension of  $g_a$ ,

$$g_a = \mu_0 C_a \simeq \frac{H}{m} \text{ A m} = \frac{V s / A}{m} \text{ A m} = V s = \text{Wb} \quad (11)$$

correctly corresponds to that for the magnetic charge. The magnetic flux  $g_a$  confined along the solenoid fans out isotropically from the end point  $\mathbf{r}_1$ . As seen in Fig. 2, a thin, semi-infinite solenoid can be viewed as a magnetic monopole located at the end.

#### IV. MAGNETIC FORCE ACTING ON A SEMI-INFINITE SOLENOID

From Eq. (2), we see that the Lorentz force acting on a tiny loop current  $\mathbf{m}$  placed at  $\mathbf{r}$  in a magnetic field  $\mathbf{B}$  is

$$\begin{aligned} \mathbf{F}_m &= \int dv' \mathbf{J}_m(\mathbf{r}') \times \mathbf{B}(\mathbf{r} + \mathbf{r}') \\ &= (\mathbf{m} \times \nabla) \times \mathbf{B}(\mathbf{r}). \end{aligned} \quad (12)$$

Using Eq. (4) and the conditions for magnetic field:  $\nabla \cdot \mathbf{B} = 0$  (divergence-free) and  $\nabla \times (\mu_0^{-1} \mathbf{B}) = 0$  (rotation-free), the expression can be modified as

$$\mathbf{F}_m = (\mathbf{m} \cdot \nabla) \mathbf{B}(\mathbf{r}), \quad (13)$$

which is suitable for line-integral. Equation (12) can also be modified as  $\mathbf{F}_m = \nabla(\mathbf{m} \cdot \mathbf{B})$  with the divergence-free condition only. The rotation-free condition is satisfied only when the right hand side of the Maxwell-Ampère equation,  $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t$ , vanishes.

Thus the magnetic force acting on a line element  $d\mathbf{l}_b$  at  $\mathbf{r}_b$  is

$$d\mathbf{F} = (C_b d\mathbf{l}_b \cdot \nabla_b) \mathbf{B}(\mathbf{r}_b), \quad (14)$$

where  $C_b = \kappa_b m$  and  $\nabla_b = \partial / \partial \mathbf{r}_b$ . Integration along a curve  $L_b$  yields the total force acting on the solenoid;

$$\begin{aligned} \mathbf{F} &= \int_{L_b} d\mathbf{F} = C_b \int_{L_b} (d\mathbf{l}_b \cdot \nabla_b) \mathbf{B} \\ &= C_b [\mathbf{B}(\mathbf{r}_3) - \mathbf{B}(\mathbf{r}_4)], \end{aligned} \quad (15)$$

where  $\mathbf{r}_4$  and  $\mathbf{r}_3$  are the initial and end points of  $L_b$ , respectively. For semi-infinite cases  $[\mathbf{B}(\mathbf{r}_4 = \infty) = 0]$ , we have

$$\mathbf{F} = C_b \mathbf{B}(\mathbf{r}_3) = g_b \mathbf{H}(\mathbf{r}_3), \quad (16)$$

with  $\mathbf{H} = \mu_0^{-1} \mathbf{B}$ . This is equal to the force for a point magnetic charge  $g_b = \mu_0 C_b$  placed at  $\mathbf{r}_3$ .

It is surprising that the sum of the forces acting on each part of the solenoid can be represented in terms only of the magnetic field  $\mathbf{B}(\mathbf{r}_3)$  at the end point  $\mathbf{r}_3$ . This is because the partial force is proportional to the (vector) gradient of the magnetic field.

#### V. COULOMB'S LAW BETWEEN TWO THIN, SEMI-INFINITE SOLENOIDS

Now we can calculate the force (16) on solenoid b by the field (9) generated by solenoid a;

$$\begin{aligned} \mathbf{F}_{b \leftarrow a} &= C_b \mathbf{B}(\mathbf{r}_3) = C_b \mu_0 \mathbf{H}(\mathbf{r}_3) \\ &= \mu_0 C_b C_a \mathbf{G}_1(\mathbf{r}_3 - \mathbf{r}_1). \end{aligned} \quad (17)$$

This equation corresponds to Coulomb's law for magnetic charges  $g_a$  at  $\mathbf{r}_1$  and  $g_b$  at  $\mathbf{r}_3$ :

$$\mathbf{F}_{3 \leftarrow 1} = \frac{g_a g_b}{\mu_0} \mathbf{G}_1(\mathbf{r}_3 - \mathbf{r}_1) = \frac{g_a g_b}{4\pi \mu_0} \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3}. \quad (18)$$

These two equations are exactly the same but the former is for the total (integrated) force between currents of solenoids and the latter is for the force between magnetic point charges.

It is interesting that the force is independent of the paths of either solenoids as long as their end points and strengths are kept constant. We should note that the derivation of the magnetic Coulomb law is not straightforward. The properties of static, source-free fields must be utilized.

#### VI. TORQUE ON A SEMI-INFINITE SOLENOID

Now we are interested in the torque due to the integrated magnetic force on a thin, semi-infinite solenoid. The torque  $\mathbf{N}_m$  with respect to the origin exerted on a loop current  $\mathbf{m}$  located at  $\mathbf{r}$  is<sup>4</sup>

$$\begin{aligned} \mathbf{N}_m &= \int dv' (\mathbf{r} + \mathbf{r}') \times [\mathbf{J}_m(\mathbf{r}') \times \mathbf{B}(\mathbf{r} + \mathbf{r}')] \\ &= \mathbf{r} \times \mathbf{F}_m + \int dv' [\mathbf{B}(\mathbf{r} + \mathbf{r}') \times \mathbf{J}_m(\mathbf{r}')] \times \mathbf{r}' \\ &= \mathbf{N}_m^{(0)} + \mathbf{N}_m^{(1)}, \end{aligned} \quad (19)$$

where the first term corresponds to the torque due to the total force  $\mathbf{F}_m$  with respect to the origin and the second term corresponds to the torque with respect to the center of the loop current. Using Eq. (13), the former can be modified as

$$\begin{aligned} \mathbf{N}_m^{(0)} &= \mathbf{r} \times \mathbf{F}_m = \mathbf{r} \times (\mathbf{m} \cdot \nabla) \mathbf{B} \\ &= (\mathbf{m} \cdot \nabla)(\mathbf{r} \times \mathbf{B}) - [(\mathbf{m} \cdot \nabla) \mathbf{r}] \times \mathbf{B} \\ &= (\mathbf{m} \cdot \nabla)(\mathbf{r} \times \mathbf{B}) - \mathbf{m} \times \mathbf{B}, \end{aligned} \quad (20)$$

with  $(\mathbf{m} \cdot \nabla) \mathbf{r} = \mathbf{m}$ . The latter can be simplified as

$$\begin{aligned} \mathbf{N}_m^{(1)} &= \int dv' [\mathbf{B}(\mathbf{r} + \mathbf{r}') \times (-\mathbf{m} \times \nabla') \delta^3(\mathbf{r}')] \times \mathbf{r}' \\ &= \mathbf{r}' \times [(\mathbf{m} \times \nabla') \times \mathbf{B}(\mathbf{r} + \mathbf{r}')] \Big|_{\mathbf{r}'=0} \\ &\quad + [\mathbf{B}(\mathbf{r} + \mathbf{r}') \times (\mathbf{m} \times \nabla')] \times \mathbf{r}' \Big|_{\mathbf{r}'=0} \\ &= 0 + [(\mathbf{B} \cdot \nabla') \mathbf{m} - (\mathbf{B} \cdot \mathbf{m}) \nabla'] \times \mathbf{r}' \Big|_{\mathbf{r}'=0} \\ &= \mathbf{m} \times \mathbf{B}, \end{aligned} \quad (21)$$

where we have utilized  $\nabla' \times \mathbf{r}' = 0$  with  $\nabla' = \partial/\partial \mathbf{r}'$ . Finally, we have

$$\mathbf{N}_m = \mathbf{N}_m^{(0)} + \mathbf{N}_m^{(1)} = (\mathbf{m} \cdot \nabla)[\mathbf{r} \times \mathbf{B}(\mathbf{r})]. \quad (22)$$

The torque on a line element  $d\mathbf{l}_b$  at  $\mathbf{r}_b$  is

$$d\mathbf{N} = C_b(d\mathbf{l}_b \cdot \nabla_b)[\mathbf{r}_b \times \mathbf{B}(\mathbf{r}_b)], \quad (23)$$

and the integration along the semi-infinite solenoid  $L_b$  terminated at  $\mathbf{r}_3$  yields

$$\begin{aligned} \mathbf{N} &= \int_{L_b} d\mathbf{N} = C_b \int_{L_b} (d\mathbf{l}_b \cdot \nabla_b)[\mathbf{r}_b \times \mathbf{B}(\mathbf{r}_b)] \\ &= C_b \mathbf{r}_3 \times \mathbf{B}(\mathbf{r}_3) = \mathbf{r}_3 \times \mathbf{F}, \end{aligned} \quad (24)$$

where  $\mathbf{F}$  is the force (16) on the solenoid. Surprisingly, the torque exerted on a thin, semi-infinite solenoid coincides with that for a point magnetic charge  $g_b = \mu_0 C_b$  placed at  $\mathbf{r}_3$ .

## VII. DISCUSSION

In this paper we only deal with thin solenoids. Solenoids with finite cross-section can be represented as a bundle of thin solenoids. Similarly a permanent magnet can naturally be modeled as a bundle of thin solenoids. The degree of magnetization of magnets or magnetized objects is characterized by the quantity called macroscopic *magnetization*,  $M \stackrel{\sim}{=} \text{A/m}$ , which is defined as volume density of magnetic moments,  $m \stackrel{\sim}{=} \text{A m}^2$ . The magnetization can also be represented as area-density of thin solenoids. We remember that a thin solenoid is characterized by the current moment,  $C \stackrel{\sim}{=} \text{A m}$ . Bundling  $\nu$  solenoids per unit area ( $\stackrel{\sim}{=} / \text{m}^2$ ), we can create the magnetization  $M = \nu C$ .

In conclusion, we have shown that the force between thin, semi-infinite solenoids obeys the Coulomb law that is for the equivalent magnetic point charges placed at each end of solenoid. We have also shown that the torque exerted on the solenoid in a magnetic field coincides with that for a corresponding magnetic point charge. It is convenient to introduce magnetic charges or poles because the magnetic Coulomb law can easily be applied for forces between them. But we should remember that without the justification given in this paper it is just a rough and ready method.

## Acknowledgments

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## APPENDIX: CURRENT DENSITY AND FIELD OF A TINY CURRENT LOOP

The current density (2) for a tiny current loop can be derived as follows. We consider a parallelogram defined

by a pair of small vectors,  $\mathbf{a}$  and  $\mathbf{b}$ . The center is located at the origin and current  $I$  is circulating along the edge. The current distribution on the four segments can be approximately represented as

$$\begin{aligned} \mathbf{J}_{I(\mathbf{a} \times \mathbf{b})}(\mathbf{r}) &= I [\mathbf{a} \delta^3(-\mathbf{b}/2) + \mathbf{b} \delta^3(\mathbf{a}/2) \\ &\quad - \mathbf{a} \delta^3(\mathbf{b}/2) - \mathbf{b} \delta^3(-\mathbf{a}/2)] \end{aligned} \quad (A.1)$$

In the limit of  $|\mathbf{a}|, |\mathbf{b}| \rightarrow 0$  with  $\mathbf{m} = I(\mathbf{a} \times \mathbf{b})$  being kept constant, it approaches

$$\begin{aligned} \mathbf{J}_m(\mathbf{r}) &= I [-\mathbf{a}(\mathbf{b} \cdot \nabla) \delta^3(\mathbf{r}) + \mathbf{b}(\mathbf{a} \cdot \nabla) \delta^3(\mathbf{r})] \\ &= [-I(\mathbf{a} \times \mathbf{b}) \times \nabla] \delta^3(\mathbf{r}) \\ &= (-\mathbf{m} \times \nabla) \delta^3(\mathbf{r}), \end{aligned} \quad (A.2)$$

and Eq. (2) is obtained. The derivative of the delta function  $\nabla \delta^3(\mathbf{r})$  serves as a differential operator when it is integrated together with other functions as in 1D cases:  $\int dx f(x)(d/dx)\delta(x) = -(df/dx)(0)$ . For example, the magnetic force on  $\mathbf{m}$  can be calculated as follows;

$$\begin{aligned} \mathbf{F}_m &= \int dv (-\mathbf{m} \times \nabla) \delta^3(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \\ &= \int dv [-\nabla \delta^3(\mathbf{m} \cdot \mathbf{B}) + \mathbf{m}(\nabla \delta^3 \cdot \mathbf{B})] \\ &= [\nabla(\mathbf{m} \cdot \mathbf{B}) - \mathbf{m}(\nabla \cdot \mathbf{B})]_{\mathbf{r}=0} \\ &= [(\mathbf{m} \times \nabla) \times \mathbf{B}](0). \end{aligned} \quad (A.3)$$

Next we derive Eq. (6) from Maxwell's equations. The Ampère law for an infinitesimal loop current,  $\nabla \times \mathbf{H} = \mathbf{J}_m$  can be written as

$$\nabla \times (\mathbf{H}(\mathbf{r}) - \mathbf{m} \delta^3(\mathbf{r})) = 0. \quad (A.4)$$

We know that a rotation-free field can be represented as a gradient of some scalar field  $\phi_m(\mathbf{r})$  as

$$\mathbf{H}(\mathbf{r}) - \mathbf{m} \delta^3(\mathbf{r}) = -\nabla \phi_m(\mathbf{r}). \quad (A.5)$$

Taking the divergence of each side, we have

$$(\nabla \cdot \mathbf{m}) \delta^3(\mathbf{r}) = \nabla^2 \phi_m(\mathbf{r}), \quad (A.6)$$

where  $\nabla \cdot (\mu_0 \mathbf{H}) = 0$  has been used. Comparing it with

$$\nabla^2 (\nabla \cdot \mathbf{m}) G_0(\mathbf{r}) = -(\nabla \cdot \mathbf{m}) \delta^3(\mathbf{r}), \quad (A.7)$$

which is obtained from  $\nabla^2 G_0(\mathbf{r}) = -\delta^3(\mathbf{r})$ , we find the solution of (A.6) to be

$$\phi_m(\mathbf{r}) = -(\nabla \cdot \mathbf{m}) G_0(\mathbf{r}). \quad (A.8)$$

With Eq. (A.5) and  $\nabla G_0 = -\mathbf{G}_1$  we have Eq. (6):

$$\mathbf{H}(\mathbf{r}) = -(\mathbf{m} \cdot \nabla) \mathbf{G}_1(\mathbf{r}) + \mathbf{m} \delta^3(\mathbf{r}), \quad (A.9)$$

which can be confirmed to satisfy the Maxwell equations.

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(1952).

<sup>6</sup> In practice, a solenoid is made as a helix of a conducting wire not as a stack of closed wire loops. The difference in current distribution, which can be represented as an additional current along the solenoid, can be made arbitrarily small by reducing the wire current and increasing the number of winding correspondingly.

<sup>7</sup> The relation  $A \stackrel{D}{\sim} B$  means  $A$  and  $B$  are dimensionally equivalent and is normally written as  $[A] = [B]$ .